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Weakly nonlinear internal waves in a two-fluid system

By WOOYOUNG CHOI AND ROBERTO CAMASSA
Theoretical Division and Center for Nonlinear Studies,
Los Alamos National Laboratory, Los Alamos, NM 87545, USA

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We derive general evolution equations for two-dimensional weakly nonlinear waves at the free surface in a system of two fluids of different densities. The thickness of the upper fluid layer is assumed to be small compared with the characteristic wavelength, but no restrictions are imposed on the thickness of the lower layer. We consider the case of a free upper boundary for its relevance in applications to ocean dynamics problems and the case of a non-uniform rigid upper boundary for applications to atmospheric problems. For the special case of shallow water, the new set of equations reduces to the Boussinesq equations for two-dimensional internal waves, whilst, for great and infinite lower-layer depth, we can recover the well-known Intermediate Long Wave and Benjamin–Ono models, respectively, for one-dimensional uni-directional wave propagation. Some numerical solutions of the model for one-dimensional waves in deep water are presented and compared with the known solutions of the uni-directional model. Finally, the effects of finite-amplitude slowly varying bottom topography are included in a model appropriate to the situation when the dependence on one of the horizontal coordinates is weak.

1. Introduction

In this paper, we consider weakly nonlinear internal waves in a two-fluid system with external forcing on the upper boundary which can be either free or rigid. The nonlinear evolution equations we derive govern, in their most general form, two-dimensional waves in a fluid of arbitrary depth.

Weakly nonlinear models for the evolution of internal waves have been extensively studied in the past. Among these, for uni-directional waves, the Korteweg–de Vries (KdV) equation for shallow water (Benjamin 1966) and the Intermediate Long Wave (ILW) equation for a fluid of finite depth including the Benjamin–Ono (BO) equation for deep water are well known (Benjamin 1967; Davis & Acrivos 1967; Ono 1975; Joseph 1977; Kubota, Ko & Dobbs 1978). On the other hand, models for two-dimensional waves have been derived only for the case of shallow water (the Boussinesq equations) although equations for weakly two-dimensional (but uni-directional) waves, in the so-called Kadomtsev & Petviashvili (1970) class, have been proposed for deep water (Ablowitz & Segur 1980).

All these well-known models exhibit many interesting features, including, for example solitary wave solutions of permanent form. However, the restrictions used in the course of the derivation of these model equations in principle limit their application to more general problems. The most severe limitation is possibly the fact that each of these models is valid only for a certain depth. Let h_{10} and h_{20} be the undisturbed depth of the upper and lower layers, respectively, and let L be a characteristic wavelength. The KdV and the Boussinesq models are valid for $h_{10}/L \ll 1$ and $h_{20}/h_{10} = O(1)$ while the ILW equation is for $h_{10}/L \ll 1$ and $h_{20}/h_{10} \gg 1$. Therefore, there is no theory in between the KdV equation and the ILW equation which can cover the whole range of the ratio h_{20}/h_{10}. Moreover, since all previous models, with the exception of the Boussinesq equations, are for uni-directional waves (or weakly two-dimensional waves with a preferred direction of propagation), general wave propagation cannot be properly described by these models. This happens, for instance, whenever reflected waves need to be taken into account, like in the case of internal waves propagating over a non-uniform sea bed in the ocean or over a hill in the atmosphere. Also, the uni-directional equations model the propagation of internal wave modes only, and nonlinear interaction of waves from different modes such as the interaction between surface and internal waves is neglected. It is therefore desirable to have a general model valid for two-dimensional waves in a fluid of arbitrary and non-uniform depth for real applications. This model should still afford the remarkable simplification over the original Euler equations achieved by the previously known models, yet it should be able to handle the more realistic situations mentioned above.

Recently, for the case of a homogeneous fluid layer, much progress has been made in this direction. Evolution equations for surface waves correct up to the third-order non-linearity in wave slope for a fluid of finite depth have been derived by Matsuno (1992) for one-dimensional waves and Choi (1995) for two-dimensional waves. These equations are general enough to comprise most of the known nonlinear evolution equations for surface waves in the appropriate limits. For the case of internal waves, Matsuno (1993*a*) has recently derived a set of equations for one-dimensional waves in a two-layer fluid of arbitrary but uniform depth. In this paper, we extend the theory of two-dimensional surface waves to internal waves in a non-homogeneous medium and discuss its various limits.

The physical set-ups we are interested in are sketched in figure 1. Both (*a*) free and (*b*) rigid upper boundaries are considered in the analysis. Having in mind applications to the dynamics of the thin thermocline near the upper free surface in the ocean, we assume the depth of the upper layer to be small compared with a characteristic wavelength; however, we make no assumptions on the depth of the lower fluid. We consider a topographical disturbance on the rigid lid for the case of rigid upper boundary, while an applied pressure on the free surface is chosen as an external forcing for the free upper boundary case. The configuration depicted in (*c*), which can model flow over a mountain in the atmosphere, is a special case of the rigid-lid case in (*b*) when the lower fluid is assumed to be infinitely deep and the direction of gravity is reversed. The separate analysis for non-uniform sea bed sketched in figure 1(*d*) is made for weakly two-dimensional bi-directional waves.

Starting with the governing equations presented in §2, we derive a set of equations in §3 for the upper layer. Assuming small h_{10}/L, we obtain the Green–Naghdi (GN) equations (Green & Naghdi 1976), by using a systematic asymptotic expansion method, for waves of arbitrary amplitude which can be further reduced to the Boussinesq equations for weakly nonlinear waves. With the evolution equations for the lower layer derived in §4, we then obtain the complete set of equations for the general case of free upper boundary, and equations for the rigid-lid situation can be readily obtained as a special case. Since the evolu-

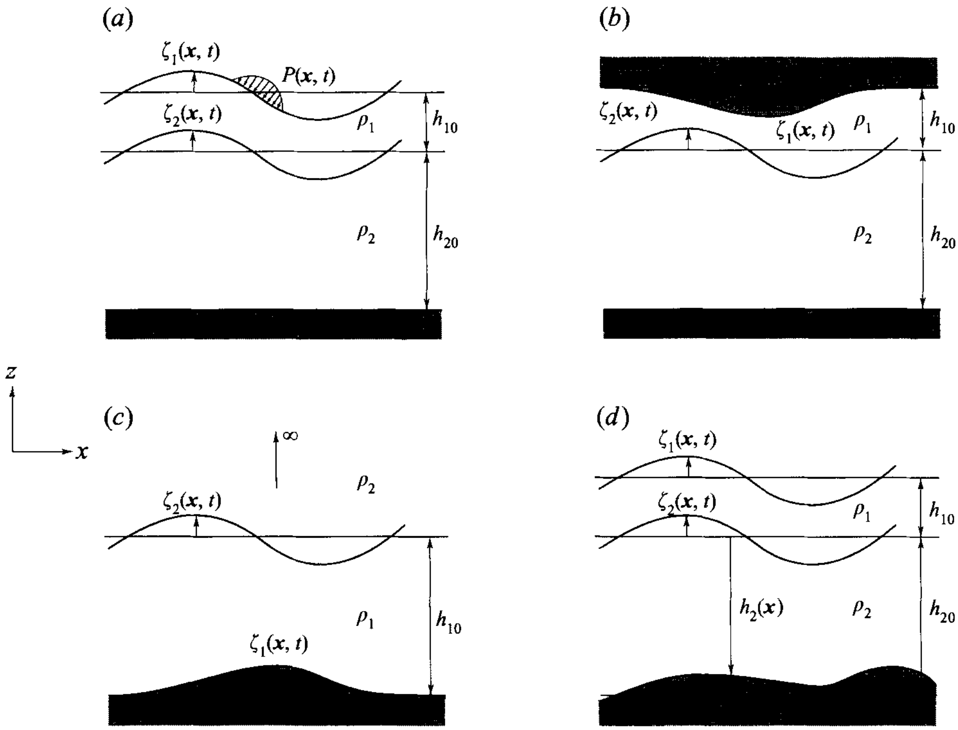

FIGURE 1. Sketches of typical set-ups of interest for the two-fluid system under consideration.

tion equations derived in this paper are valid for arbitrary water depth, all known nonlinear models can be recovered as special cases. In §5, we discuss the limit of great lower-layer depth, in which two-dimensional versions of the ILW or BO equation are derived, while the opposite, shallow water, limit is considered in the Appendix. We also present some numerical solutions of the bi-directional model for one-dimensional waves in deep water. Moreover, in this section we show that higher-order equations for the case of great lower-layer depth can be found with little modification from the new set of equations for arbitrary depth. Finally, to study the long-time behaviour of internal waves propagating from relatively deep to shallow water, we derive in §6 evolution equations which include effects of a slowly varying sea bed under the assumption of weakly two-dimensional wave motion.

2. Basic equations

The Cartesian coordinates $(\boldsymbol{x}, z) = (x, y, z)$ are introduced with origin at the interface of two fluids of different densities, ρ_1 for the upper fluid and ρ_2 for the lower fluid. The velocity components $(\boldsymbol{u}_i, w_i) = (u_i, v_i, w_i)$ and the pressure p_i for inviscid and incompressible fluids satisfy the continuity equation and the Euler equations:

$$\nabla \cdot \boldsymbol{u}_i + w_{iz} = 0, \tag{2.1}$$

$$\boldsymbol{u}_{it} + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i + w_i \boldsymbol{u}_{iz} = -\nabla p_i / \rho_i, \tag{2.2}$$

$$w_{it} + \boldsymbol{u}_i \cdot \nabla w_i + w_i w_{iz} = -p_{iz}/\rho_i - g, \tag{2.3}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$ and subscripts with respect to coordinates or time stand for partial differentiation. These equations apply to both upper and lower fluids, respectively for $i = 1$ and $i = 2$. The kinematic and dynamic boundary conditions at the upper free surface ($z = h_{10} + \zeta_1$) are given by

$$\zeta_{1t} + (\boldsymbol{u}_1 \cdot \nabla)\zeta_1 = w_1, \qquad p_1 = p_a + P(\boldsymbol{x}, t) \qquad \text{at} \quad z = h_{10} + \zeta_1(\boldsymbol{x}, t), \tag{2.4}$$

where h_{10} is the thickness of the undisturbed upper layer, $\zeta_1(\boldsymbol{x}, t)$ is a displacement of the upper free surface, p_a is an atmospheric pressure (taken to be zero) and $P(\boldsymbol{x}, t)$ is the external pressure applied to the free surface (see figure 1*a*). At the interface ($z = \zeta_2$), the boundary conditions are

$$\zeta_{2t} + (\boldsymbol{u}_1 \cdot \nabla)\zeta_2 = w_1, \qquad \zeta_{2t} + (\boldsymbol{u}_2 \cdot \nabla)\zeta_2 = w_2, \qquad p_1 = p_2 \qquad \text{at} \quad z = \zeta_2(\boldsymbol{x}, t). \tag{2.5}$$

At the known bottom topography in the lower fluid ($z = -h_2$), the kinematic boundary condition is given by

$$(\boldsymbol{u}_2 \cdot \nabla)h_2 + w_2 = 0 \qquad \text{at} \quad z = -h_2(\boldsymbol{x}). \tag{2.6}$$

In the following analysis, we will use the layer-mean equations (Wu 1981) obtained by integrating (2.1)–(2.2) across the vertical layer of the upper fluid, $\zeta_2 \leqslant z \leqslant h_{10} + \zeta_1$, and imposing the boundary conditions (2.4)–(2.5),

$$\eta_{1t} + \nabla \cdot (\eta_1 \overline{\boldsymbol{u}_1}) = 0, \qquad \eta_1 = h_{10} + \zeta_1 - \zeta_2, \tag{2.7}$$

$$(\eta_1 \overline{\boldsymbol{u}_1})_t + \nabla \cdot (\eta_1 \overline{\boldsymbol{u}_1 \boldsymbol{u}_1}) = -\eta_1 \overline{\nabla p_1}/\rho_1, \tag{2.8}$$

where $\eta_1 = h_{10} + \zeta_1 - \zeta_2$ is the thickness of the perturbed upper layer and

$$\overline{f}(\boldsymbol{x}, t) \equiv \frac{1}{\eta_1} \int_{\zeta_2}^{h_{10}+\zeta_1} f(\boldsymbol{x}, z, t)\mathrm{d}z. \tag{2.9}$$

Note that these layer-mean equations (2.7)–(2.8) are exact for inviscid and incompressible fluids and can apply to both rotational and irrotational flows. They imply local conservation laws and integration with respect to x from $-\infty$ to ∞ reveals the global conservation laws for mass and horizontal momentum in hydrodynamics. Of course, since this system is not closed due to new additional unknowns such as $\overline{\boldsymbol{u}_1 \boldsymbol{u}_1}$, it is useful in practice only if a closure scheme, like the one to be discussed in the following section, can be devised.

3. Evolution equations for the thin upper layer

We assume that the upper layer is thin, i.e. we can introduce a small parameter $\epsilon = h_{10}/L \ll 1$, where L is characteristic length in the horizontal direction, for example wavelength. Then, from (2.1) with $i = 1$, we have

$$\frac{w_1}{|\boldsymbol{u}_1|} = O(h_{10}/L) = O(\epsilon) \ll 1. \tag{3.1}$$

First, we non-dimensionalize the physical variables for the upper fluid as

$$\left.\begin{array}{c} \boldsymbol{x} = L\boldsymbol{x}^*, \quad z = h_{10}z^*, \quad t = (L/U_0)t^*, \\ p_1 = (\rho_1 U_0^2)p_1^*, \quad (\zeta_1, \zeta_2) = h_{10}(\zeta_1^*, \zeta_2^*), \quad (\boldsymbol{u}_1, w_1) = U_0(\boldsymbol{u}_1^*, \epsilon w_1^*), \end{array}\right\} \tag{3.2}$$

where U_0 is a typical velocity, which we choose as $U_0 = (gh_{10})^{1/2}$. After dropping the asterisks for dimensionless variables, equations (2.7) and (2.8) remain unaltered in form except $\rho_1 = 1$ in (2.8) while the vertical momentum equation (2.3) can be written as

$$p_{1z} = -1 - \epsilon^2 \left[w_{1t} + (\boldsymbol{u}_1 \cdot \nabla) w_1 + w_1 w_{1z} \right]. \tag{3.3}$$

From (3.3), we can expand $f = (\boldsymbol{u}_1, w_1, p_1)$ as

$$f(\boldsymbol{x}, z, t) = f^{(0)} + \epsilon^2 f^{(1)} + O(\epsilon^4), \tag{3.4}$$

and notice that *no assumption on amplitude is imposed* in this expansion.

By substituting (3.4) into (3.3) and imposing the dynamic boundary condition in (2.4), the leading-order pressure $p_1{}^{(0)}$ is obtained as

$$p_1{}^{(0)} = -(z - \zeta_1) + P(\boldsymbol{x}, t), \tag{3.5}$$

and, by substituting (3.4) and (3.5) into (2.2), it follows that $\boldsymbol{u}_1{}^{(0)}$ is independent of the vertical coordinate, $\boldsymbol{u}_1{}^{(0)} = \boldsymbol{u}_1{}^{(0)}(\boldsymbol{x}, t)$, if the initial condition

$$\partial_z \boldsymbol{u}_1{}^{(0)} = 0 \qquad \text{at} \quad t = 0 \tag{3.6}$$

is chosen. This condition (3.6) can be fulfilled in some special cases of interest, for example (i) irrotational flow, (ii) a flow only with a strong z-directional vorticity (such as a rotating fluid for geophysical application), (iii) a flow with continuous weak shear or weak stratification, in the sense that the flow is almost uniform in the z-direction or the fluid is almost homogeneous in density, and so on. In this paper we are interested in a two-fluid system without any initial vorticity, and so we assume that the flow in each fluid is irrotational. We can now obtain $w_1{}^{(0)}$ from (2.1) with the kinematic boundary condition in (2.5) as

$$w_1{}^{(0)} = -(\nabla \cdot \boldsymbol{u}_1{}^{(0)})(z - \zeta_2) + D_1 \zeta_2, \tag{3.7}$$

where

$$D_1 \zeta_2 = \zeta_{2t} + (\boldsymbol{u}_1{}^{(0)} \cdot \nabla) \zeta_2. \tag{3.8}$$

From (3.4), (3.7) and the condition of zero horizontal vorticity given by

$$\partial_z \boldsymbol{u}_1 = \epsilon^2 \nabla w_1, \tag{3.9}$$

it is easy to show that, if condition (3.6) holds,

$$\eta_1 \overline{\boldsymbol{u}_1 \boldsymbol{u}_1} = \eta_1 \overline{\boldsymbol{u}_1}\, \overline{\boldsymbol{u}_1} + O(\epsilon^4), \tag{3.10}$$

and the layer-mean horizontal momentum equation (2.8) in dimensionless form can be written as

$$\overline{\boldsymbol{u}_1}_t + (\overline{\boldsymbol{u}_1} \cdot \nabla) \overline{\boldsymbol{u}_1} = -\overline{\nabla p_1} + O(\epsilon^4). \tag{3.11}$$

At $O(\epsilon^2)$, from (3.3), the equation for p_1 is given by

$$\begin{aligned} \partial_z p_1{}^{(1)} &= - \left[\partial_t w_1{}^{(0)} + (\boldsymbol{u}_1{}^{(0)} \cdot \nabla) w_1{}^{(0)} + w_1{}^{(0)} \partial_z w_1{}^{(0)} \right] \\ &= G_1(\boldsymbol{x}, t)(z - \zeta_2) + F_1(\boldsymbol{x}, t), \end{aligned} \tag{3.12}$$

where we have used the expression (3.7) for $w_1{}^{(0)}$ and F_1 and G_1 are defined as

$$G_1(\boldsymbol{x}, t) = \nabla \cdot \partial_t \boldsymbol{u}_1{}^{(0)} + \boldsymbol{u}_1{}^{(0)} \cdot \nabla \left(\nabla \cdot \boldsymbol{u}_1{}^{(0)} \right) - \left(\nabla \cdot \boldsymbol{u}_1{}^{(0)} \right)^2, \qquad F_1(\boldsymbol{x}, t) = -D_1{}^2 \zeta_2. \tag{3.13}$$

By imposing the dynamic boundary condition in (2.4), $p_1{}^{(1)}$ can be written as

$$p_1{}^{(1)}(\boldsymbol{x}, z, t) = \tfrac{1}{2} G_1(\boldsymbol{x}, t) \left[(z - \zeta_2)^2 - (h_{10} + \zeta_1 - \zeta_2)^2 \right] + F_1(\boldsymbol{x}, t) \left[(z - \zeta_2) - (h_{10} + \zeta_1 - \zeta_2) \right]. \tag{3.14}$$

From (3.5) and (3.14), the right-hand-side term of the horizontal momentum equation (3.11) is given by

$$\begin{aligned}\overline{\nabla p_1} &= -\overline{\nabla\left(p_1{}^{(0)} + \epsilon^2 p_1{}^{(1)}\right)} + O(\epsilon^4)\\ &= \nabla\zeta_1 + \nabla P\\ &\quad -\epsilon^2\left[\frac{1}{\eta_1}\nabla\left(\tfrac{1}{3}\eta_1{}^3 G_1 + \tfrac{1}{2}\eta_1{}^2 F_1\right) + \left(\tfrac{1}{2}\eta_1 G_1 + F_1\right)\nabla\zeta_2\right] + O(\epsilon^4), \qquad (3.15)\end{aligned}$$

and, from (2.7) and (3.11) with (3.15), the final equations for the upper fluid can be written, in dimensional form, as

$$\eta_{1_t} + \nabla\cdot(\eta_1\overline{u_1}) = 0, \qquad \eta_1 = h_{10} + \zeta_1 - \zeta_2, \qquad (3.16)$$

$$\overline{u_1}_t + \overline{u_1}\cdot\nabla\overline{u_1} + g\nabla\zeta_1 = -\nabla P/\rho_1 + \frac{1}{\eta_1}\nabla\left(\tfrac{1}{3}\eta_1{}^3 G_1 + \tfrac{1}{2}\eta_1{}^2 F_1\right) + \left(\tfrac{1}{2}\eta_1 G_1 + F_1\right)\nabla\zeta_2 + O(\epsilon^4). \qquad (3.17)$$

Here $u_1{}^{(0)}$ is replaced by $\overline{u_1}$ in the expression for G_1 given by (3.13) in this approximation. While the first equation (3.16) implying conservation of mass is exact, the momentum equation (3.17) has an error term of $O(\epsilon^4)$. This is the same system of equations as the one derived by Green & Naghdi (1976) for open channel flows based on the director-sheet model, when we take $\zeta_2(x,t)$ as a known bottom topography. From our derivation, it can be seen that this set of equations is valid for long waves of finite amplitude comparable to the thickness of the layer. It is also apparent that this system can be regarded as the higher-order version of the shallow water equations† since the leading-order dispersive terms of $O(\epsilon^2)$ neglected in the classical shallow water equations (Lamb 1932, §187) have now been included.

Up to now, we have not made any assumption on the amplitude of the dependent variables and the only small parameter we have introduced is the aspect ratio ϵ. When the depth of the lower layer h_{20} is comparable to h_{10} (in other words, for shallow water), equations for the lower fluid can be readily written from (3.16)–(3.17) without further analysis (see the Appendix). However, in order to find evolution equations for a lower fluid of *arbitrary* depth, we will consider weak nonlinear effects from now on. We introduce another small parameter α defined by

$$\alpha = a/h_{10} \ll 1, \qquad (3.18)$$

where a is a characteristic wave amplitude, so that equations (3.16)–(3.17) reduce, in this regime of weakly nonlinear waves, to

$$\eta_{1_t} + \nabla\cdot(\eta_1\overline{u_1}) = 0, \qquad \eta_1 = h_{10} + \zeta_1 - \zeta_2, \qquad (3.19)$$

$$\overline{u_1}_t + \overline{u_1}\cdot\nabla\overline{u_1} + g\nabla\zeta_1 = -\nabla P/\rho_1 + \nabla\left(\tfrac{1}{3}h_{10}{}^2\nabla\cdot\overline{u_1}_t - \tfrac{1}{2}h_{10}\zeta_{2tt}\right) + O(\alpha\epsilon^4, \alpha^2\epsilon^2). \qquad (3.20)$$

Here we have assumed

$$\zeta_1/h_{10} = O(\zeta_2/h_{10}) = O(\overline{u_1}/U_0) = O(\alpha), \qquad (3.21)$$

and $\epsilon^2 \leqslant \alpha \leqslant \epsilon$ in order to take the leading-order nonlinear and dispersive effects into account. For balance between these two effects, the scaling between α and ϵ is known to be $\alpha = O(\epsilon^2)$ for shallow water and $\alpha = O(\epsilon)$ for deep water (as we shall see

† Recently, Wei *et al.* (1995) also derived a fully nonlinear Boussinesq model for shallow water which reduces to the Green–Naghdi model when the depth-mean horizontal velocity is used as dependent variable rather than the horizontal velocity at a prescribed height.

later). Notice that (3.19) and (3.20) are the Boussinesq equations for surface waves in shallow water when ζ_2 is regarded as a known bottom topography (Whitham 1974; Wu 1987).

For a two-fluid system, the required matching condition across the interface between the two fluids is the pressure continuity given by (2.5), which becomes

$$
\begin{aligned}
p_2(\boldsymbol{x},\zeta_2,t) &= p_1(\boldsymbol{x},\zeta_2,t) \\
&= -\rho_1 g(\zeta_2-\zeta_1) + P - \rho_1\left(\tfrac{1}{2}\eta_1{}^2 G_1 + \eta_1 F_1\right) + O(\epsilon^4) \\
&= -\rho_1 g(\zeta_2-\zeta_1) + P - \rho_1\left(\tfrac{1}{2}h_{10}{}^2\nabla\cdot\overline{\boldsymbol{u}_1}_t - h_{10}\zeta_{2tt}\right) + O(\alpha\epsilon^4,\alpha^2\epsilon^2),
\end{aligned}
\tag{3.22}
$$

after (3.5) and (3.14) are used to evaluate $p_1(\boldsymbol{x},\zeta_2,t)$ and, in the last expression, we impose the weakly nonlinear assumption (3.21). We now have the complete set of equations governing the dynamics of the upper fluid, under the thin layer and weak nonlinearity assumptions: the kinematic equation (3.19), the dynamic equation (3.20) and the matching condition (3.22).

4. Evolution equations for the lower fluid

To derive evolution equations for the lower fluid of arbitrary depth, we follow the method developed in Choi (1995) for surface waves. We present the essential points and refer the reader to Choi (1995) for details. Throughout this section, we assume a flat bottom at $z=-h_{20}$ in the lower fluid. The effects of a non-uniform sea bed will be considered in §6.

For the lower fluid of arbitrary depth, it is more appropriate to use the physical variables evaluated at the interface (at $z=\zeta_2$) rather than the layer-mean value used for the thin upper layer. By substituting $z=\zeta_2$ into the Euler equations (2.2)–(2.3) with $i=2$, the dynamic equation for the horizontal velocity vector of the lower fluid evaluated at the interface is given by

$$
\widetilde{\boldsymbol{u}_2}_t + \widetilde{\boldsymbol{u}_2}\cdot\nabla\widetilde{\boldsymbol{u}_2} + g\nabla\zeta_2 = -\nabla\widetilde{p_2}/\rho_2 - (D_2{}^2\zeta_2)\nabla\zeta_2,
\tag{4.1}
$$

where

$$
\widetilde{\boldsymbol{u}_2} = \boldsymbol{u}_2(\boldsymbol{x},\zeta_2,t), \qquad \widetilde{p_2} = p_2(\boldsymbol{x},\zeta_2,t), \qquad D_2\zeta_2 = \zeta_{2t} + \widetilde{\boldsymbol{u}_2}\cdot\nabla\zeta_2.
\tag{4.2}
$$

In deriving (4.1), we have used the chain rule for differentiation and the kinematic boundary condition (2.5). The resulting dynamic equation (4.1) is exact. For weak nonlinearity (3.21) and under the assumption $\widetilde{\boldsymbol{u}_2}/\overline{\boldsymbol{u}_1} = O(1)$, (4.1) is approximately

$$
\widetilde{\boldsymbol{u}_2}_t + \widetilde{\boldsymbol{u}_2}\cdot\nabla\widetilde{\boldsymbol{u}_2} + g\nabla\zeta_2 = -\nabla\widetilde{p_2}/\rho_2 + O(\alpha^2\epsilon^2),
\tag{4.3}
$$

where we have dropped $D_2{}^2\zeta_2\nabla\zeta_2 \sim \zeta_{2tt}\nabla\zeta_2$ from (4.1) for consistency with the approximations leading to (3.19)–(3.20). After imposing the matching condition (3.22), the dynamic equation (4.3) becomes

$$
\begin{aligned}
&\widetilde{\boldsymbol{u}_2}_t + \widetilde{\boldsymbol{u}_2}\cdot\nabla\widetilde{\boldsymbol{u}_2} + (1-\rho_r)g\nabla\zeta_2 + \rho_r g\nabla\zeta_1 \\
&\qquad = -\nabla P/\rho_2 + \rho_r\nabla\left(\tfrac{1}{2}h_{10}{}^2\nabla\cdot\overline{\boldsymbol{u}_1}_t - h_{10}\zeta_{2tt}\right) + O(\alpha\epsilon^4,\alpha^2\epsilon^2),
\end{aligned}
\tag{4.4}
$$

where we assume $\rho_r = (\rho_1/\rho_2) < 1$ for stable stratification. We now have determined the dynamic equation for the lower fluid consistent with the level of approximation of the dynamic and kinematic evolution equations for the upper fluid. The remaining step is to find the kinematic equation for the lower fluid.

Since we have assumed that the flow is irrotational, we can introduce the velocity potential $\phi(\boldsymbol{x},z,t)$ for the lower fluid which, from the continuity equation (2.1) for

$i = 2$, solves the following boundary value problem:

$$\left(\nabla^2 + \partial_z^2\right)\phi = 0 \qquad \text{for} \quad -h_{20} \leqslant z \leqslant \zeta_2(\boldsymbol{x}, t), \tag{4.5}$$

with, from (2.5)–(2.6), the kinematic boundary conditions given by

$$\phi_z - \nabla\zeta_2 \cdot \nabla\phi = \zeta_{2t} \qquad \text{at} \quad z = \zeta_2(\boldsymbol{x}, t), \tag{4.6}$$

$$\phi_z = 0 \qquad \text{at} \quad z = -h_{20}, \tag{4.7}$$

where $(\nabla\phi, \phi_z) = (\boldsymbol{u}_2, w_2)$ and the depth of the lower fluid h_{20} is constant.

By expanding (4.6) about $z = 0$ in Taylor series as

$$\phi_z = \zeta_{2t} + \nabla \cdot (\zeta_2 \widetilde{\boldsymbol{u}_2}) + O(\alpha^3\epsilon^3) \qquad \text{at} \quad z = 0, \tag{4.8}$$

and using the Fourier transform method, we can find the formal solution of (4.5) correct up to $O(\alpha^2\epsilon^2)$ in the strip $-h_{20} < z < 0$. This yields (see Choi (1995) for details)

$$\zeta_{2t} + \nabla \cdot (\zeta_2 \widetilde{\boldsymbol{u}_2}) = \boldsymbol{T} \cdot \left[\nabla\phi(\boldsymbol{x}, 0, t)\right], \tag{4.9}$$

where $\boldsymbol{T} \cdot [\boldsymbol{v}]$ is defined by

$$\boldsymbol{T} \cdot [\boldsymbol{v}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\nabla \cdot \boldsymbol{v})\, K(|\boldsymbol{x} - \boldsymbol{x}'|; h_{20})\, \mathrm{d}x'\, \mathrm{d}y', \tag{4.10}$$

with

$$K(|\boldsymbol{x}|; h_{20}) = -\int_0^{\infty} \tanh(kh_{20})\, J_0(k|\boldsymbol{x}|)\, \mathrm{d}k, \tag{4.11}$$

and $J_0(x)$ is the zeroth-order Bessel function. By expanding $\widetilde{\boldsymbol{u}_2}$ about $z = 0$, the expression for $\nabla\phi(\boldsymbol{x}, 0, t)$ can be found, in terms of ζ_2 and $\widetilde{\boldsymbol{u}_2}$, as

$$\widetilde{\boldsymbol{u}_2} = \nabla\phi\Big|_{z=\zeta_2} = \nabla\phi(\boldsymbol{x}, 0, t) + \zeta_2\nabla\zeta_{2t} + O(\alpha^3\epsilon^3). \tag{4.12}$$

Substituting (4.12) into (4.9) yields the kinematic equation for the lower fluid, correct up to the second order in the wave slope parameter $\alpha\epsilon$,

$$\zeta_{2t} + \nabla \cdot (\zeta_2 \widetilde{\boldsymbol{u}_2}) - \boldsymbol{T} \cdot \left[\widetilde{\boldsymbol{u}_2} - \zeta_2\nabla\zeta_{2t}\right] = O(\alpha^3\epsilon^3). \tag{4.13}$$

Finally, with (3.19), (3.20), (4.4) and (4.13), we have the complete set of equations for the displacement of the upper free surface ζ_1, the displacement of the interface ζ_2, the depth-mean velocity across the thin upper layer $\overline{\boldsymbol{u}_1}$ and the velocity of the lower fluid evaluated at the interface $\widetilde{\boldsymbol{u}_2}$. In dimensional form, the complete system is

$$\eta_{1t} + \nabla \cdot (\eta_1 \overline{\boldsymbol{u}_1}) = 0, \qquad \eta_1 = h_{10} + \zeta_1 - \zeta_2, \tag{4.14}$$

$$\overline{\boldsymbol{u}_1}_t + \overline{\boldsymbol{u}_1} \cdot \nabla\overline{\boldsymbol{u}_1} + g\nabla\zeta_1 = -\nabla P/\rho_1 + \nabla\left(\tfrac{1}{3}{h_{10}}^2\nabla \cdot \overline{\boldsymbol{u}_1}_t - \tfrac{1}{2}h_{10}\zeta_{2tt}\right), \tag{4.15}$$

$$\widetilde{\boldsymbol{u}_2}_t + \widetilde{\boldsymbol{u}_2} \cdot \nabla\widetilde{\boldsymbol{u}_2} + (1 - \rho_r)g\nabla\zeta_2 + \rho_r g\nabla\zeta_1 = -\nabla P/\rho_2 + \rho_r\nabla\left(\tfrac{1}{2}{h_{10}}^2\nabla \cdot \overline{\boldsymbol{u}_1}_t - h_{10}\zeta_{2tt}\right), \tag{4.16}$$

$$\zeta_{2t} + \nabla \cdot (\zeta_2 \widetilde{\boldsymbol{u}_2}) - \boldsymbol{T} \cdot \left[\widetilde{\boldsymbol{u}_2} - \zeta_2\nabla\zeta_{2t}\right] = 0. \tag{4.17}$$

By substituting the leading-order equations such as $\zeta_{2t} = \zeta_{1t} + h_{10}\nabla \cdot \overline{\boldsymbol{u}_1} + O(\alpha^2)$ from (4.14) into the higher-order terms containing time derivatives, we can obtain various other forms of equations asymptotically equivalent to (4.14)–(4.17), all of which have relative errors of $O(\alpha\epsilon^4, \alpha^2\epsilon^2)$ with $\epsilon^2 \leqslant \alpha \leqslant \epsilon$. In this respect, notice that the double time derivative ζ_{2tt} in (4.15)–(4.16) could represent a short-hand notation for $\boldsymbol{T}\cdot[\widetilde{\boldsymbol{u}}_{2t}]$.

For one-dimensional waves, $\boldsymbol{T}\cdot[\widetilde{\boldsymbol{u}}_2]$ can be reduced, after performing the integration in (4.10) with respect to y, to $\mathscr{T}[\widetilde{u}_2]$ defined by

$$\mathscr{T}[\widetilde{u}_2] = -\frac{1}{2h_{20}}\mathscr{P}\int_{-\infty}^{\infty}\frac{\widetilde{u}_2(x',t)}{\sinh\left[(\pi/2h_{20})(x'-x)\right]}\,\mathrm{d}x', \tag{4.18}$$

where $\mathscr{P}$ stands for the integration as Cauchy principal value. From (4.14)–(4.17), the equations for one-dimensional waves are given, in dimensional form, by

$$\eta_{1t} + (\eta_1\overline{u_1})_x = 0, \qquad \eta_1 = h_{10} + \zeta_1 - \zeta_2, \tag{4.19}$$

$$\overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x + g\zeta_{1x} = -P_x/\rho_1 + \tfrac{1}{3}h_{10}{}^2\overline{u_1}_{xxt} - \tfrac{1}{2}h_{10}\zeta_{2xtt}, \tag{4.20}$$

$$\widetilde{u}_{2t} + \widetilde{u}_2\widetilde{u}_{2x} + (1-\rho_r)g\zeta_{2x} + \rho_r g\zeta_{1x} = -P_x/\rho_2 + \rho_r\left(\tfrac{1}{2}h_{10}{}^2\overline{u_1}_{xxt} - h_{10}\zeta_{2xtt}\right), \tag{4.21}$$

$$\zeta_{2t} + (\widetilde{u}_2\zeta_2)_x - \mathscr{T}\left[\widetilde{u}_2 - \zeta_2\zeta_{2xt}\right] = 0. \tag{4.22}$$

The dispersion relation for the linearized version of equations (4.19)–(4.22) is determined from the following equation for the wave frequency ω:

$$\begin{aligned}&\omega^4\left[\left(1+\tfrac{1}{3}k^2h_{10}{}^2\right)/\tanh(kh_{20}) + \rho_r kh_{10}\left(1+\tfrac{1}{12}k^2h_{10}{}^2\right)\right]\\ &-\omega^2 gk\left[kh_{10}/\tanh(kh_{20}) + \left(1+\tfrac{1}{3}k^2h_{10}{}^2\right)\right] + (1-\rho_r)g^2k^3h_{10} = 0,\end{aligned} \tag{4.23}$$

where k is the wavenumber. The linear dispersion relation (4.23) is consistent with the small-kh_{10} limit of the full linear dispersion relation for a system of two fluids of finite depth (Lamb 1932, §231). Since (4.23) is a quadratic equation in ω^2, two independent wave modes exist and the evolution equations (4.14)–(4.17) (or (4.19)–(4.22)) support both surface and internal wave modes. For example, when taking the limit of $kh_{20}\to\infty$, we have the surface wave mode $c^2 = \omega^2/k^2 = g/k$ and the internal wave mode $c^2 = gh_{10}(1-\rho_r)$ at leading order for small kh_{10}.

As ρ_r approaches 1, the leading-order terms in (4.14)–(4.17) imply that ζ_1/ζ_2 tends to zero (Lamb 1932, §231). In this case, the rigid-lid approximation can be used to study internal wave modes only. For the rigid-lid case, equations (4.14)–(4.17) are still the governing equations when P is regarded as the unknown pressure at the rigid lid and ζ_1 is considered as a known perturbation to the flat rigid lid. By eliminating the unknown pressure P from (4.19)–(4.22) for one-dimensional waves, we obtain the evolution equations for the rigid-lid case as

$$\zeta_{2t} - \left[(h_{10} + \zeta_1 - \zeta_2)\overline{u_1}\right]_x = \zeta_{1t}, \tag{4.24}$$

$$\overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x - \left(\frac{1}{\rho_r} - 1\right)g\zeta_{2x} = \frac{1}{\rho_r}\left(\widetilde{u}_{2t} + \widetilde{u}_2\widetilde{u}_{2x}\right) + \tfrac{1}{3}h_{10}{}^2\overline{u_1}_{xxt} + \tfrac{1}{2}h_{10}\zeta_{1xtt}, \tag{4.25}$$

$$\zeta_{2t} + (\widetilde{u}_2\zeta_2)_x - \mathscr{T}\left[\widetilde{u}_2 - \zeta_2\zeta_{2xt}\right] = 0, \tag{4.26}$$

where we have used $\zeta_{2t} = \zeta_{1t} + h_{10}\overline{u_1}_x + O(\alpha^2)$ in (4.25).

The evolution equations for one-dimensional waves (4.19)–(4.22) (or (4.24)–(4.26)) can also be obtained from equations derived by Matsuno (1993*a*) by a change of dependent variables (for example, by using the velocity at the free surface instead of the layer-mean velocity for the upper fluid). The form presented here has the advantage of being more compact.

Since the new sets of equations can be applied to a fluid of arbitrary depth, they can be reduced to various known equations as special cases. For shallow water ($h_{20} = O(h_{10})$), (4.14)–(4.17) can be reduced to the Boussinesq equations and to the KdV equation for uni-directional waves (see the Appendix). For great lower-layer depth ($h_{20} \gg h_{10}$), they can be reduced to the two-dimensional version of the ILW-equation or the BO-equation as we will show in the following section.

5. Equations for a fluid of great lower-layer depth

For great lower-layer depth ($h_{20} \gg h_{10}$), we assume that a characteristic length in the vertical direction for the lower fluid is comparable to L, the typical horizontal length scale. It can be noticed, from (2.1) for $i = 2$, that

$$w_2/|\boldsymbol{u}_2| = O(1). \tag{5.1}$$

By continuity of the normal velocity at $z = \zeta_2$ in (2.5)

$$\boldsymbol{u}_1 \cdot \nabla\zeta_2 - w_1 = \boldsymbol{u}_2 \cdot \nabla\zeta_2 - w_2 \qquad \text{at} \quad z = \zeta_2, \tag{5.2}$$

the leading-order approximation with $\nabla\zeta_2 = O(\alpha\epsilon)$ gives

$$w_2/|\boldsymbol{u}_1| = O(w_1/|\boldsymbol{u}_1|) = O(\epsilon) \qquad \text{at } z = \zeta_2, \tag{5.3}$$

where we have used (3.1) in the last expression. From (5.1) and (5.3), we then have the following order estimates:

$$\frac{|w_2|\Big|_{z=\zeta_2}}{|\boldsymbol{u}_1|\Big|_{z=\zeta_2}} = O(\epsilon), \quad \frac{|\boldsymbol{u}_2|\Big|_{z=\zeta_2}}{|\boldsymbol{u}_1|\Big|_{z=\zeta_2}} = O(\epsilon). \tag{5.4}$$

These estimates can be understood by keeping in mind the fact that the fluid motion in the lower layer, excited by the motion of the upper layer, is necessarily small compared to the upper-layer motion owing to the relatively larger domain (and mass) of the lower fluid.

Guided by (3.21) and (5.4) (also following Benjamin 1967), we assume

$$\overline{\boldsymbol{u}_1}/U_0 = O(\zeta_1/h_{10}) = O(\zeta_2/h_{10}) = O(\alpha), \qquad \alpha = O(\epsilon), \qquad \widetilde{\boldsymbol{u}_2}/U_0 = O(\alpha\epsilon) = O(\epsilon^2), \tag{5.5}$$

where the second scaling gives the balance between nonlinear and dispersive effects in a fluid of great depth.

By neglecting all terms of order higher than $O(\epsilon^2)$, with $\alpha = O(\epsilon)$, the set of equations (4.14)–(4.17) can be simplified to

$$\eta_{1t} + \nabla \cdot (\eta_1 \overline{\boldsymbol{u}_1}) = 0, \qquad \eta_1 = h_{10} + \zeta_1 - \zeta_2, \tag{5.6}$$

$$\overline{\boldsymbol{u}_1}_t + \overline{\boldsymbol{u}_1} \cdot \nabla \overline{\boldsymbol{u}_1} + g\nabla\zeta_1 = -\nabla P/\rho_1, \tag{5.7}$$

$$\widetilde{\boldsymbol{u}_2}_t + (1 - \rho_r) g \nabla\zeta_2 + \rho_r g \nabla\zeta_1 = -\nabla P/\rho_2, \tag{5.8}$$

$$\zeta_{2t} - \boldsymbol{T} \cdot [\widetilde{\boldsymbol{u}_2}] = 0. \tag{5.9}$$

The set of four equations (5.6)–(5.9) is the complete system governing the dynamics of $(\zeta_1, \zeta_2, \overline{\boldsymbol{u}_1}, \widetilde{\boldsymbol{u}_2})$ in a fluid of great lower-layer depth with either free or rigid upper boundary. For the case of free upper boundary, P is known and ζ_1 is unknown and vice versa for the case of a rigid lid. In deep water ($h_{20} \to \infty$), $\boldsymbol{T} \cdot [\widetilde{\boldsymbol{u}_2}]$ reduces to $\boldsymbol{H} \cdot [\widetilde{\boldsymbol{u}_2}]$ defined by

$$\boldsymbol{H} \cdot [\widetilde{\boldsymbol{u}_2}] = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\nabla \cdot \widetilde{\boldsymbol{u}_2}}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}\boldsymbol{x}'. \tag{5.10}$$

For atmospheric applications (see figure 1*c*), we need to replace g by $-g$ with $h_{20} \to \infty$ in the formulation for the rigid lid, and assume $\rho_r = (\rho_1/\rho_2) > 1$ for stable stratification.

5.1. *The model equations for uni-directional waves*

To derive uni-directional models, we non-dimensionalize all physical variables with respect to h_{10} and $U_0 = (gh_{10})^{1/2}$ as

$$(x, y) = h_{10}(\hat{x}, \hat{y}), \quad t = (h_{10}/U_0)\hat{t}, \tag{5.11}$$

$$(\overline{\boldsymbol{u}_1}, \zeta_1, \zeta_2, \widetilde{\boldsymbol{u}_2}, P) = (U_0\hat{\boldsymbol{u}}_1, h_{10}\hat{\zeta}_1, h_{10}\hat{\zeta}_2, U_0\hat{\boldsymbol{u}}_2, \rho_1 {U_0}^2 \hat{P}). \tag{5.12}$$

For uni-directional waves with weak dependence on y, we substitute, from (5.5), the following expansions into (5.6)–(5.9):

$$\xi = \epsilon(\hat{x} - c_0\hat{t}), \qquad Y = \epsilon^{3/2}\hat{y}, \qquad \tau = \epsilon^2\hat{t}, \tag{5.13}$$

$$f(\xi, Y, \tau) = \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots, \qquad f = (\hat{u}_1, \hat{\zeta}_1, \hat{\zeta}_2), \tag{5.14}$$

$$\hat{v}_1(\xi, Y, \tau) = \epsilon^{3/2}{\hat{v}_1}^{(1)} + \epsilon^{5/2}{\hat{v}_1}^{(2)} + \cdots, \tag{5.15}$$

$$\hat{u}_2(\xi, Y, \tau) = \epsilon^2{\hat{u}_2}^{(1)} + \cdots, \qquad \hat{P} = \epsilon^2 \Pi(\xi, Y, \tau), \tag{5.16}$$

where c_0 is a linear wave speed to be determined and the external forcing P is assumed to move with a speed close to c_0 in (5.16). At first order, we have

$$c_0^2 = (1 - \rho_r), \qquad {\hat{u}_1}^{(1)} = -\frac{1}{c_0}\left(\frac{1}{\rho_r} - 1\right){\hat{\zeta}_2}^{(1)}, \tag{5.17}$$

$${\hat{\zeta}_1}^{(1)} = -\left(\frac{1}{\rho_r} - 1\right){\hat{\zeta}_2}^{(1)}, \qquad \partial_\xi \hat{v}_1^{(1)} = -\frac{g}{c_0}\left(\frac{1}{\rho_r} - 1\right)\partial_Y {\hat{\zeta}_2}^{(1)}. \tag{5.18}$$

In (5.17), only internal wave modes are found even though both internal and surface wave modes exist in the original system (5.6)–(5.9). We exclude fast modes (surface wave modes) in the expansion (5.13) by choosing waves with a constant speed of propagation, such as, for instance, solitary waves.

At $O(\epsilon^2)$, by imposing the solvability condition, we can obtain the evolution equation for $\zeta_2(x, y, t)$ which can be written in dimensional form as

$$\left[\zeta_{2t} + c_0\zeta_{2x} - \frac{3c_0}{2\rho_r h_{10}}\zeta_2\zeta_{2x} - \frac{\rho_r h_{10}}{2}\widetilde{u_{2x}}(x, y, 0, t)\right]_x + \frac{c_0}{2}\zeta_{2yy} = \frac{\rho_r c_0}{2\rho_1 g}P_{xx}, \tag{5.19}$$

where, from (5.9) and after replacing ζ_{2t} by $-c_0\zeta_{2x}$, $\widetilde{u_{2x}}$ is given by

$$\widetilde{u_{2x}} = -c_0\mathscr{T}^{-1}\left[\zeta_{2xx}\right], \tag{5.20}$$

and $\mathscr{T}^{-1}$, the inverse transform of $\mathscr{T}$, is defined as

$$\mathscr{T}^{-1}[f] = \frac{1}{2h_{20}}\mathscr{P}\int_{-\infty}^{\infty} f(x') \coth\left[\frac{\pi}{2h_{20}}(x' - x)\right] \mathrm{d}x'. \tag{5.21}$$

For one-dimensional waves, (5.19) becomes the forced ILW-equation:

$$\zeta_{2t} + c_0\zeta_{2x} - \frac{3c_0}{2\rho_r h_{10}}\zeta_2\zeta_{2x} + \frac{c_0\rho_r h_{10}}{2}\mathscr{T}^{-1}[\zeta_{2xx}] = \frac{\rho_r c_0}{2\rho_1 g}P_x, \tag{5.22}$$

where the positive (negative) c_0 is taken for the right-going (left-going) waves. For deep water ($h_{20} \to \infty$), $\mathscr{T}^{-1}[\zeta_2]$ in (5.22) reduces to the Hilbert transform $\mathscr{H}[\zeta_2]$ defined by

$$\mathscr{H}[\zeta_2] = \mathscr{P} \int_{-\infty}^{\infty} \frac{\zeta_2(x', y, t)}{x' - x} \mathrm{d}x'. \tag{5.23}$$

With (5.20) and (5.23), the evolution equation (5.19) (without the forcing term) is the equation first derived by Ablowitz & Segur (1980). When the y-dependence is dropped, (5.19) is the forced Benjamin–Ono (fBO) equation.

For the rigid-lid case, the expansions for P and ζ_1 are

$$\frac{P}{\rho_1 U_0{}^2} = \epsilon \hat{P}^{(1)} + \epsilon^2 \hat{P}^{(2)} + \cdots, \qquad \frac{\zeta_1}{h_{10}} = \epsilon^2 \hat{\zeta}_1(\xi, Y, \tau), \tag{5.24}$$

and, by following procedures similar to the free boundary case, the evolution equation for uni-directional waves with weak dependence on y can be found to be

$$\left(\zeta_{2t} + c_0 \zeta_{2x} - \frac{3c_0}{2h_{10}} \zeta_2 \zeta_{2x} + \frac{c_0 h_{10}}{2\rho_r} \mathscr{T}^{-1}[\zeta_{2xx}] \right)_x + \frac{c_0}{2} \zeta_{2yy} = -\frac{c_0}{2} \zeta_{1xx}, \tag{5.25}$$

where the linear speed c_0 is given by

$$c_0{}^2 = g h_{10} \left(\frac{1}{\rho_r} - 1 \right), \tag{5.26}$$

and a disturbance at the flat rigid lid, $\zeta_1(x,t) = O(\epsilon^2)$, is assumed to move with speed close to c_0. Comparing (5.25) with (5.19), we can see that the pressure forcing P in the case of a free upper boundary and the topographical disturbance ζ_1 in the rigid-lid case have the same effect in the uni-directional weakly nonlinear analysis, a situation which is similar to the case of pressure forcing on the free surface versus bottom topography in the shallow water problem (Wu 1987).

5.2. *Travelling waves in deep water*

In this section, the bi-directional model for one-dimensional waves in deep water ($h_{20} \to \infty$) with a uniform rigid lid ($\zeta_1 = 0$) will be studied numerically. From (5.6)–(5.9), the bi-directional model in this case is

$$\zeta_{2t} - \left[(h_{10} - \zeta_2) \overline{u_1} \right]_x = 0, \tag{5.27}$$

$$\overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x - \left(\frac{1}{\rho_r} - 1 \right) g \zeta_{2x} = \frac{h_{10}}{\rho_r} \mathscr{H}[\overline{u_1}_{xt}], \tag{5.28}$$

where $\widetilde{u_2} = \mathscr{H}[\zeta_{2t}]$ from (5.9) and $\zeta_{2t} = h_{10} \overline{u_1}_x$ from (5.27) at the leading-order approximation have been used to obtain the right-hand-side term of (5.28).

The uni-directional model corresponding to (5.27)–(5.28) is the BO equation given by (5.25) with (5.23) and $\zeta_1 = 0$:

$$\zeta_{2t} + c_0 \zeta_{2x} + \gamma_1 \zeta_2 \zeta_{2x} + \gamma_2 \mathscr{H}[\zeta_{2xx}] = 0, \tag{5.29}$$

where the linear wave speed c_0 is given by (5.26) and

$$\gamma_1 = -\frac{3c_0}{2h_{10}}, \qquad \gamma_2 = \frac{c_0 h_{10}}{2\rho_r}. \tag{5.30}$$

This BO model has the periodic solution of wavelength λ (Benjamin 1967):

$$\zeta_{2p}(X) = \frac{A}{1 - B\cos\left(2\pi X/\lambda\right)}, \tag{5.31}$$

where

$$X = x-(c_0+\delta)t, \qquad \delta = \frac{a\gamma_1}{4}, \qquad A = \frac{32\pi^2\gamma_2{}^2}{a\gamma_1{}^2\lambda^2}, \qquad B = \left[1 - \left(\frac{8\pi\gamma_2}{a\gamma_1\lambda}\right)^2\right]^{1/2}. \tag{5.32}$$

This is a two-parameter family (a, λ) of periodic wave solutions and, as $\lambda \to \infty$, it reduces to the solitary wave solution (Benjamin 1967):

$$\zeta_{2s}(X) = \frac{al^2}{X^2 + l^2}, \tag{5.33}$$

where

$$|l| = \frac{4\gamma_2}{a\gamma_1}, \qquad \delta = \frac{\gamma_2}{|l|} = \frac{a\gamma_1}{4}. \tag{5.34}$$

As pointed out by Benjamin (1967), the displacement of the interface is always downward ($a < 0$), since γ_1 and γ_2 have different signs for both right- and left-going waves. Also, from (5.34), the solitary wave is a supercritical phenomenon ($\delta > 0$).

After we non-dimensionalize all variables with respect to c_0 and h_{10} (say $c_0 = 1$, $h_{10} = 1$), we solve (5.27)–(5.28) with periodic boundary conditions in space by using the pseudospectral method (Fletcher 1990) with the number of Fourier modes $N \geqslant 32$ and a second-order time integration scheme such as the leap-frog method with time increment $\Delta t = 0.2$. For all computations, the density ratio $\rho_r = 0.9$ is chosen and, for the solitary wave solution, large λ ($\lambda = 400$) is taken.

To test our numerical codes, we study the propagation of a single free solitary wave. Since no explicit solitary wave solution is known for this bi-directional model, we take as initial conditions $\zeta_2(x,0) = \zeta_{2p}(x)$ given by (5.31), $\overline{u_1}(x,0) = -\zeta_{2p}(x)$ for a right-going wave at the leading-order approximation and compare numerical solutions with (5.31) at $t = 200$. For small amplitude ($a = -0.05, -0.1$), the periodic solution of the BO equation approximates well the solution of the bi-directional model as shown in figure 2.

Next, by taking the following initial conditions for ζ_2 and $\overline{u_1}$:

$$\zeta_2(x,0) = \zeta_{2p}(x), \qquad \overline{u_1}(x,0) = 0 \qquad \text{at } t = 0, \tag{5.35}$$

we solve (5.27)–(5.28) with $a = -0.05$. Since there is no preferred direction of wave propagation due to (5.35), the initial hump at $x = 0$ evolves into two identical solitary waves propagating in opposite directions. Owing to the periodic boundary condition, they collide with waves propagating from adjacent computational domains. As shown in figure 3, the numerical solutions show 'clean' interaction with no discernible phase shift after collision. We also study the head-on collision of two identical solitary waves of amplitude $a = -0.05$, originally located at $x = \pm x_0$ but with opposite directions of propagation:

$$\zeta_2(x,0) = \zeta_{2p}^+(x-x_0)+\zeta_{2p}^-(x+x_0), \qquad \overline{u_1}(x,0) = \zeta_{2p}^+(x-x_0)-\zeta_{2p}^-(x+x_0) \qquad \text{at } t = 0, \tag{5.36}$$

where $x_0 = 100$ is taken in the computations. As shown in figure 4, the numerical solutions at $t = 200$ show a small phase shift compared with the two-solitary-wave solution given by (5.31) without interaction. This phase shift seems to persist even

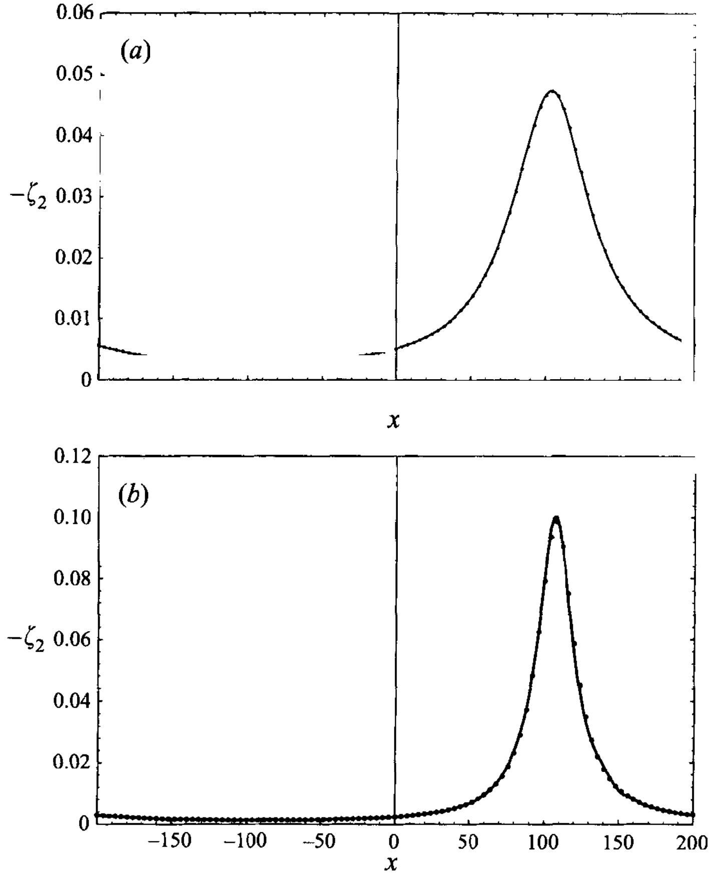

FIGURE 2. Comparison of the numerical solution for a travelling wave of the bi-directional model (5.27)–(5.28) at $t = 200$ with the travelling wave solution of the BO equation given by (5.31) with $\lambda = 400$, initially located at $x = -100$: ——, numerical solution; $\cdots$, solution of the BO equation. (*a*) $a = -0.05$ (*b*) $a = -0.1$.

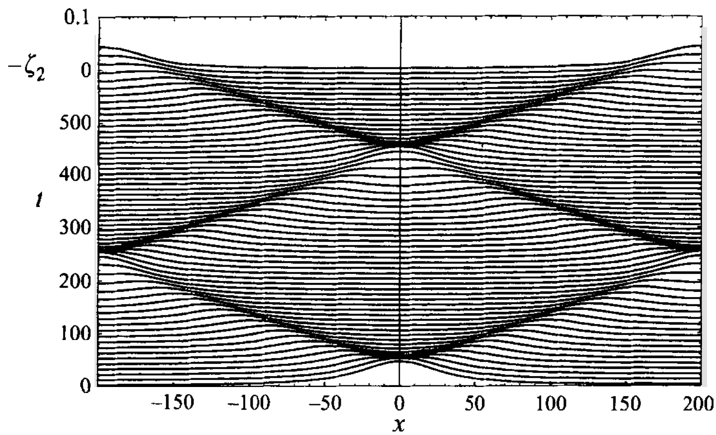

FIGURE 3. Evolution of the interface from the initial condition $\overline{u_1}(x,0) = 0$ and $\zeta_2(x,0) = \zeta_{2p}(x)$ given by (5.31) with $a = -0.05$ and $\lambda = 400$.

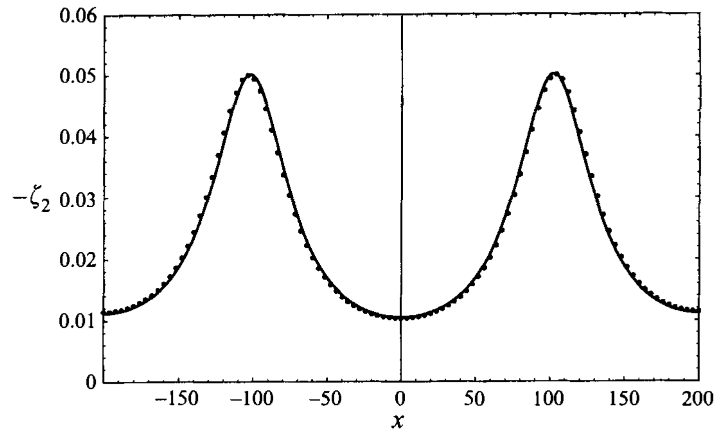

FIGURE 4. Head-on collision of two solitary waves. Numerical solutions (——) are compared with the linear superposition of the two-solitary-wave solution (···), given by (5.31) with $\lambda = 400$, at $t = 200$. Initially two identical solitary waves of amplitude $a = -0.05$ propagating in opposite directions were located at $x = \pm 100$.

when we use a higher-order time integration scheme, or increase the number of Fourier modes or the spacing between initial waves.

For finite $h_{20}(\gg h_{10})$, (5.27)–(5.28) are still the governing equations for bi-directional waves when the Hilbert transform $\mathscr{H}$ is replaced by $\mathscr{T}^{-1}$ defined in (5.21). In this case calculations similar to the infinite-depth case can be carried out by using the periodic steady wave solution of the ILW-equation obtained by Miloh (1990).

5.3. *The higher-order equations for the case of a rigid lid*

In order to derive equations (5.6)–(5.9), we have neglected terms of order higher than $O(\epsilon^2)$ with $\alpha = O(\epsilon)$ from (4.14)–(4.17). However, without any further analysis, higher-order equations correct up to $O(\epsilon^3)$ can be easily found for the case of great lower-layer depth ($h_{20} \gg h_{10}$). Since the terms neglected in (4.14)–(4.17) are $O(\epsilon^4)$ with the scaling in (5.5), equations (4.14)–(4.17) are the consistent higher-order equations for $h_{20} \gg h_{10}$ if we drop $\tilde{\boldsymbol{u}}_2 \cdot \nabla \tilde{\boldsymbol{u}}_2$, which is $O(\epsilon^4)$, from (5.5).

For the rigid-lid case, equations (4.14)–(4.17) without $\tilde{\boldsymbol{u}}_2 \cdot \nabla \tilde{\boldsymbol{u}}_2$ are also the higher-order equations for unknown pressure P and known disturbance ζ_1 at the rigid lid. However, $\zeta_1 = O(\epsilon)$ is too restrictive to model a topographic disturbance like a mountain in the atmosphere, and the more general assumption $\zeta_1/h_{10} = O(1)$ and $\zeta_{1x} = O(\epsilon)$ is desirable. The rest of this section will be devoted to the derivation of the higher-order equations modelling the effects of topographical disturbances of finite amplitude at the rigid lid for the case of $h_{20} \gg h_{10}$.

For the upper fluid, we recall that the GN equations (3.16)–(3.17) are the governing equations for arbitrary amplitudes of ζ_1 and ζ_2. By imposing $\zeta_1/h_{10} = O(1)$ and $\zeta_2/h_{10} = O(\epsilon)$ on the dynamic equation (3.17) and applying the matching condition (3.22) on (4.3), we can obtain two dynamic equations, which replace (4.15) and (4.16):

$$\overline{\boldsymbol{u}_1}_t + \overline{\boldsymbol{u}_1} \cdot \nabla \overline{\boldsymbol{u}_1} + g\nabla\zeta_1 = -\nabla P/\rho_1 + \frac{1}{h_1}\nabla\left(\tfrac{1}{3}h_1^3\nabla\cdot\overline{\boldsymbol{u}_1}_t - \tfrac{1}{2}h_1^2\zeta_{2tt}\right) + O(\epsilon^4), \tag{5.37}$$

$$\tilde{\boldsymbol{u}}_{2t} + (1-\rho_r)g\nabla\zeta_2 + \rho_r g\nabla\zeta_1 = -\nabla P/\rho_2 + \rho_r\nabla\left(\tfrac{1}{2}h_1^2\nabla\cdot\overline{\boldsymbol{u}_1}_t - h_1\zeta_{2tt}\right) + O(\epsilon^4), \tag{5.38}$$

where we assume a stationary disturbance at the rigid lid $\zeta_1 = \zeta_1(x)$, and

$$h_1(x) = h_{10} + \zeta_1(x). \tag{5.39}$$

Equations (4.14), (4.17), (5.37) and (5.38) are the higher-order system for the case of a rigid lid with a topographical disturbance $\zeta_1/h_{10} = O(1)$. For one-dimensional waves, by eliminating P from (5.37) and (5.38), we can simplify the governing equations to

$$\zeta_{2t} - \Big[(h_1 - \zeta_2)\overline{u_1}\Big]_x = 0, \tag{5.40}$$

$$\overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x - \left(\frac{1}{\rho_r} - 1\right) g\zeta_{2x} = \frac{1}{\rho_r}\mathscr{T}^{-1}[(h_1 - \zeta_2)\overline{u_1}]_{xt} - \tfrac{1}{6}h_1^2\overline{u_1}_{xxt} + \tfrac{1}{2}h_1\left(h_1\overline{u_1}_t\right)_{xx}, \tag{5.41}$$

where we have used $\zeta_{2t} = \Big[(h_1 - \zeta_2)\overline{u_1}\Big]_x$ in (5.41).

As a special case of a flat rigid lid in deep water ($h_1 = h_{10}$ and $h_{20} \to \infty$), the uni-directional model (5.40)–(5.41) can be readily obtained, by using the expansion method introduced in §5.1, as

$$\zeta_{2t} + c_0\zeta_{2x} - \frac{3c_0}{2h_{10}}\zeta_2\zeta_{2x} + \frac{c_0h_{10}}{2\rho_r}\mathscr{H}[\zeta_{2xx}] + \frac{c_0h_{10}^2}{6}\left(1 - \frac{9}{4}\frac{1}{\rho_r^2}\right)\zeta_{2xxx}$$
$$- \frac{3c_0}{8h_{10}^2}\zeta_2^2\zeta_{2x} - \frac{c_0}{2\rho_r}\left(\tfrac{9}{4}\mathscr{H}[\zeta_2\zeta_{2x}]_x + \tfrac{5}{4}\zeta_2\mathscr{H}[\zeta_{2xx}] + \zeta_{2x}\mathscr{H}[\zeta_{2x}]\right) = 0, \tag{5.42}$$

where c_0 is given by (5.26). This is the higher-order Benjamin–Ono equation derived by Matsuno (1994).

6. Effects of a slowly varying sea bed

In this section, we consider the effects of a non-uniform 'sea bed' in the lower fluid of finite depth (see figure 1*d*), which have been neglected in the previous analysis. Although the evolution equations (4.14)–(4.15) for the upper fluid and the dynamic equation (4.16) for the lower fluid remain valid for the case of non-uniform bottom, the kinematic equation for the lower fluid has to be modified. The governing equation for the velocity potential ϕ becomes (4.5) with (4.6) and, instead of (4.7), the kinematic boundary condition at $z = -h_2(x, y)$

$$\phi_z = -\nabla\phi \cdot \nabla h_2 \qquad \text{at } z = -h_2(x, y), \tag{6.1}$$

must be imposed. For a small variation from a flat bottom ($|(h_2 - h_{20})/h_{20}| \ll 1$), the Fourier transform method introduced in §4 can still be used after expanding the bottom boundary condition (6.1) about $z = -h_{20}$. However, we are interested in the case of $|(h_2 - h_{20})/h_{20}| = O(1)$ and $h_{2x} \ll 1$. In order to keep the analysis simple, we only consider weakly two-dimensional waves with $h_{2y}/h_{2x} \ll 1$ for which the non-uniform bottom at $z = -h_2(x, y)$ can be transformed into one at $z_1 = -b(y)$ via conformal mapping. The Fourier transform method can then be used by regarding y as a parameter as we shall presently see.

Let L_y be a characteristic length in the y-direction and introduce the small parameter μ defined by

$$\mu = L/L_y, \qquad \mu^2 = O(\alpha\epsilon). \tag{6.2}$$

Equation (6.2) characterizes the assumed weak y-dependence of the flow, implying $\partial_y^2 = O(\alpha\epsilon)$, so that we can introduce the slow variable $y_s = \mu y$ for the y-coordinate.

We first need a transformation that maps the (x, y_s, z)-plane into the (x_1, y_s, z_1)-plane where the bottom is independent of x_1. The mapping function from $-h_2(x, y_s) \leqslant z \leqslant 0$ to $-b(y_s) \leqslant z_1 \leqslant 0$ is given by (Matsuno 1993*b*)

$$z(x_1, z_1, y_s) = \frac{1}{2d} \int_{-\infty}^{\infty} \frac{\widehat{h_2}(x_1 + x_1', y_s) \sin(\pi z_1/b)}{\cosh(\pi x_1'/b) + \cos(\pi z_1/b)} \, \mathrm{d}x_1', \tag{6.3}$$

where $\widehat{h_2}(x_1, y_s) \equiv h_2(x(x_1, -b), y_s)$ and y_s can be regarded as a parameter. Assuming a slowly varying bottom such as

$$h_2(x, y_s) = h_2(\delta x, \delta y_s) = O(1), \qquad \delta = L/L_h = O(\alpha\epsilon), \tag{6.4}$$

where L_h is a characteristic length for the bottom variation, the leading-order term in (6.3), after we expand $\widehat{h_2}(\delta x_1 + \delta x_1', \delta y_s)$ for small $\delta x_1'$ and integrate with respect to x_1', gives

$$\frac{\partial z}{\partial z_1} = \frac{\partial x}{\partial x_1} = \left(\frac{\widehat{h_2}}{b}\right) + O(\delta^2), \qquad \frac{\partial z}{\partial x_1} = -\frac{\partial x}{\partial z_1} = \delta \left(\frac{\widehat{h_2}_{x_1}}{b}\right) z_1 + O(\delta^3), \tag{6.5}$$

where we have used the Cauchy–Riemann relations. We also have

$$\frac{\partial x_1}{\partial x} = \frac{\partial z_1}{\partial z} = \left(\frac{b}{\widehat{h_2}}\right) + O(\delta^2), \qquad \frac{\partial x_1}{\partial z} = -\frac{\partial z_1}{\partial x} = \delta \left(\frac{\widehat{h_2}_{x_1}}{\widehat{h_2}}\right) \left(\frac{b}{\widehat{h_2}}\right) z_1 + O(\delta^3), \tag{6.6}$$

and integrating $\partial x_1/\partial x$ with respect to x gives

$$x_1 = \int_0^x \frac{b(\delta y_s)}{h_2(\delta x', \delta y_s)} \mathrm{d}x' + O(\delta^2), \qquad \mathrm{d}x = (\widehat{h_2}/b)\, \mathrm{d}x_1 + O(\delta^2). \tag{6.7}$$

From (4.5)–(4.6) and (6.1), the governing equations can then be transformed into

$$\hat{\phi}_{x_1x_1} + \hat{\phi}_{z_1z_1} + \mu^2 \left(\widehat{h_2}/b\right)^2 \hat{\phi}_{y_sy_s} = 0 \qquad \text{for} \quad -b(\delta y_s) \leqslant z_1 \leqslant 0, \tag{6.8}$$

$$\hat{\phi}_{z_1} = \left(\widehat{h_2}/b\right) \hat{R}(x_1, y_s, t) \qquad \text{at} \quad z_1 = 0, \tag{6.9}$$

$$\hat{\phi}_{z_1} = 0 \qquad \text{at} \quad z_1 = -b(\delta y_s), \tag{6.10}$$

where

$$\hat{\phi} = \phi\left(x(x_1, z_1), y_s, z(x_1, z_1)\right), \tag{6.11}$$

and the errors in (6.8)–(6.10) are $O(\alpha^3\epsilon^3)$. In (6.9), $R(x, y_s, t)$ is the vertical velocity at $z = 0$ given, from (4.8) after imposing (6.2), by

$$R(x, y_s, t) = \zeta_{2t} + (\zeta_2 \widetilde{u_2})_x + O(\alpha^3\epsilon^3). \tag{6.12}$$

One way to solve (6.8)–(6.10) is to expand all dependent variables in powers of $(\alpha\epsilon)$ and solve (6.8) by taking the one-dimensional Fourier transform at each order (in the second order, the governing equation becomes the Poisson equation due to $\phi_{y_sy_s}$). A less rigorous but convenient way is to define $y_* = (b/\widehat{h_2})y$ and regard $\widehat{h_2}(\delta x_1, \delta y_s)$ and $b(\delta y_s)$ as constants since they are slowly varying. Then the same method as in Choi (1995) can be used to find the evolution equation. Following the second approach (for details, see the steps in §§4 and 5.3 of Choi 1995), we obtain the kinematic equation for weakly two-dimensional waves in the form

$$\left[h_2\left(\zeta_{2t}+(\widetilde{u_2}\zeta_2)_x-\mathscr{T}_2[\widetilde{u_2}-\zeta_2\zeta_{2xt}]\right)\right]_x-\tfrac{1}{2}\mathscr{T}_2[h_2\widetilde{v_2}_y]+\tfrac{1}{2}h_2{}^2\widetilde{u_2}_{yy}+\tfrac{1}{2}\mathscr{T}_2\mathscr{T}_2\left[h_2{}^2\widetilde{u_2}_{yy}\right]=O(\alpha^2\epsilon^2), \tag{6.13}$$

where $\mathscr{T}_2$ is given by

$$\mathscr{T}_2[g] = -\frac{1}{2h_2}\mathscr{P}\int_{-\infty}^{\infty}\frac{g(x',y,t)}{\sinh\left[\dfrac{\pi}{2}\displaystyle\int_x^{x'}\frac{dx''}{h_2(x'',y)}\right]}\,dx'. \tag{6.14}$$

By imposing the condition of irrotational flow, $\overline{u_1}_y = \overline{v_1}_x$ and $\widetilde{u_2}_y = \widetilde{v_2}_x$ at the leading order, $\overline{v_1}$ and $\widetilde{v_2}$ can be eliminated from (4.14)–(4.16) and (6.13) after using (6.2) and differentiating the equations with respect to x. The complete set of equations in dimensional form for weakly two-dimensional waves in a fluid of slowly varying depth is

$$\eta_{1xt} + (\eta_1\overline{u_1})_{xx} + h_{10}\overline{u_1}_{yy} = 0, \qquad \eta_1 = h_{10} + \zeta_1 - \zeta_2, \tag{6.15}$$

$$\overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x + g\zeta_{1x} = -P_x/\rho_1 + \tfrac{1}{3}h_{10}{}^2\overline{u_1}_{xxt} - \tfrac{1}{2}h_{10}\zeta_{2xtt}, \tag{6.16}$$

$$\widetilde{u_2}_t + \widetilde{u_2}\widetilde{u_2}_x + (1-\rho_r)g\zeta_{2x} + \rho_r g\zeta_{1x} = -P_x/\rho_2 + \tfrac{1}{2}\rho_r h_{10}{}^2\overline{u_1}_{xxt} - h_{10}\zeta_{2xtt}, \tag{6.17}$$

$$\left(h_2\zeta_{2t} - h_2\mathscr{T}_2[\widetilde{u_2}]\right)_{xx} + h_2\left[(\widetilde{u_2}\zeta_2)_x + \mathscr{T}_2[\zeta_2\zeta_{2xt}]\right]_{xx} - \tfrac{1}{2}h_2\mathscr{T}_2[\widetilde{u_2}_{yy}] + \tfrac{1}{2}h_2{}^2\left(\widetilde{u_2}_{yy} + \mathscr{T}_2\mathscr{T}_2[\widetilde{u_2}_{yy}]\right)_x = 0, \tag{6.18}$$

where $h_{2x} = O(\delta) = O(\alpha\epsilon)$ and

$$\mathscr{T}_2[h_2 f] = h_2\mathscr{T}_2[f] + O(\delta) \tag{6.19}$$

have been used to obtain (6.18) from (6.13). Since no restriction on the depth is imposed, (6.15)–(6.18) can be used to study the evolution of internal waves generated in relatively deep water and propagating into shallower water. For the rigid-lid case, we can simplify the evolution equations (6.15)–(6.18) by eliminating P as before. Notice that (6.18) can be reduced to the kinematic equation derived by Matsuno (1993*b*) for one-dimensional surface waves in a fluid of slowly varying depth by dropping the y-dependence and to the equation for weakly two-dimensional surface waves (Matsuno 1993*c*) by assuming uniform depth $h_2(x) = h_{20}$.

7. Conclusions

We have derived various sets of nonlinear evolution equations for two-dimensional waves in a two-fluid system of arbitrary depth with either a free or rigid upper boundary. The elevations of the free surface and interface between two fluids are assumed to be small compared with the thickness of the upper layer while the amplitude of topographical disturbances at the rigid lid or at the sea bed can be finite. These new equations for waves propagating in a non-homogeneous medium due to mountains or a slowly varying sea bed might be valuable for practical applications. Most of the well-known nonlinear models are shown to be recovered as special cases and other nonlinear models can be derived directly from the new set of equations by taking the limit appropriate to the problem of interest.

Our analysis is based on the assumption of the upper layer being thin compared with a characteristic wavelength, but it can be extended to the case of both layers being of finite depth. However this would considerably increase the difficulty in solving the resulting equations owing to appearance of an additional non-local operator. Although we only consider the case of two-layer fluids in this paper, the model for many layers can be easily obtained by the approach adopted here. Also, higher-order equations for a fluid of arbitrary depth can be derived with the combination of the higher-order Boussinesq equations for the upper fluid and the third-order evolution equation for the lower fluid derived in Choi (1995). The numerical scheme developed here has been used only for a few examples in deep water chosen for their simplicity, but it can be adapted to more complicated cases.

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Appendix. The coupled Green–Naghdi equations for shallow water

In §3, we have shown that the nonlinear evolution equations valid for finite-amplitude long waves are the GN equations for a thin upper fluid layer. If the depth of the lower fluid is comparable to that of the upper fluid, say $h_{20}/h_{10} = O(1)$, equations for the lower fluid corresponding to (3.16)–(3.17) can be readily written without any further analysis. In this Appendix, the slowly varying bottom at $z = -h_2(\boldsymbol{x})$ is also included in the analysis.

By replacing $(h_{10} - \zeta_2,\ \zeta_1,\ \overline{\boldsymbol{u}_1},\ P/\rho_1)$ in (3.16)–(3.17) by $(h_2,\ \zeta_2,\ \overline{\boldsymbol{u}_2},\ \widetilde{p_2}/\rho_2)$, equations for the lower fluid can be obtained as

$$\eta_{2t} + \nabla\cdot(\eta_2\overline{\boldsymbol{u}_2}) = 0, \qquad \eta_2 = \zeta_2 + h_2, \tag{A 1}$$

$$\overline{\boldsymbol{u}_2}_t + \overline{\boldsymbol{u}_2}\cdot\nabla\overline{\boldsymbol{u}_2} + g\nabla\zeta_2 = -\nabla\widetilde{p_2}/\rho_2 + \frac{1}{\eta_2}\nabla\left(\tfrac{1}{3}{\eta_2}^3G_2 + \tfrac{1}{2}{\eta_2}^2F_2\right) - \left(\tfrac{1}{2}\eta_2G_2 + F_2\right)\nabla h_2, \tag{A 2}$$

where

$$\overline{\boldsymbol{u}_2}(\boldsymbol{x},t) = \frac{1}{\eta_2}\int_{-h_2}^{\zeta_2} \boldsymbol{u}_2(\boldsymbol{x},z,t)\mathrm{d}z, \tag{A 3}$$

$$G_2(\boldsymbol{x},t) = \nabla\cdot\overline{\boldsymbol{u}_2}_t + \overline{\boldsymbol{u}_2}\cdot\nabla(\nabla\cdot\overline{\boldsymbol{u}_2}) - (\nabla\cdot\overline{\boldsymbol{u}_2})^2, \qquad F_2(\boldsymbol{x},t) = (\overline{\boldsymbol{u}_2}\cdot\nabla)^2h_2. \tag{A 4}$$

By substituting (3.22) for $\widetilde{p_2}$, which reads

$$\widetilde{p_2} = -\rho_1g(\zeta_2 - \zeta_1) + P - \rho_1\left(\tfrac{1}{2}G_1{\eta_1}^2 + F_1\eta_1\right), \tag{A 5}$$

where F_1 and G_1 are given in (3.13), into (A 2), the complete set of equations for four unknowns (ζ_1, ζ_2, $\overline{\boldsymbol{u}_1}$ and $\overline{\boldsymbol{u}_2}$) is given by (3.16), (3.17), (A 1) and (A 2) with (A 5) which are the (coupled) GN equations for two-dimensional internal waves in a two-fluid system. It can be shown that the structure of the GN equations for the one-layer case, for example energy conservation and Hamiltonian structure, carries over to the two-fluid system as well.

For weakly nonlinear waves of the Boussinesq family with $\alpha = O(\epsilon^2)$, the GN system can be further reduced to the Boussinesq equations which, for the case of a free upper boundary, can be written as

$$\eta_{1t} + \nabla\cdot(\eta_1\overline{\boldsymbol{u}_1}) = 0, \qquad \eta_1 = h_{10} + \zeta_1 - \zeta_2, \tag{A 6}$$

$$\overline{\boldsymbol{u}_1}_t + \overline{\boldsymbol{u}_1}\cdot\nabla\overline{\boldsymbol{u}_1} + g\nabla\zeta_1 = -\nabla P/\rho_1 + \tfrac{1}{3}{h_{10}}^2\nabla(\nabla\cdot\overline{\boldsymbol{u}_1}_t) - \tfrac{1}{2}h_{10}\nabla\zeta_{2tt}, \tag{A 7}$$

$$\eta_{2t} + \nabla \cdot (\eta_2 \overline{\boldsymbol{u}_2}) = 0, \qquad \eta_2 = \zeta_2 + h_2, \tag{A 8}$$

$$\overline{\boldsymbol{u}_2}_t + \overline{\boldsymbol{u}_2} \cdot \nabla \overline{\boldsymbol{u}_2} + g(1-\rho_r)\nabla\zeta_2 + g\rho_r\nabla\zeta_1 = -\nabla P/\rho_2 + \tfrac{1}{3}h_2{}^2\nabla(\nabla\cdot\overline{\boldsymbol{u}_2}_t) + \rho_r\nabla\left(\tfrac{1}{2}h_{10}{}^2\nabla\cdot\overline{\boldsymbol{u}_1}_t - h_{10}\zeta_{2tt}\right), \tag{A 9}$$

where terms neglected in this set of equations are less than $O(\alpha\epsilon^4, \alpha^2\epsilon^2)$ with the following order estimations:

$$|\overline{\boldsymbol{u}_1}|/U_0 = O(|\overline{\boldsymbol{u}_2}|/U_0) = O(\zeta_1/h_{10}) = O(\zeta_2/h_{10}) = O(\alpha), \tag{A 10}$$

$$h_2/h_{10} = O(1), \qquad \nabla h_2 = O(\alpha\epsilon). \tag{A 11}$$

For uni-directional waves, equations (A 6)–(A 9) can be reduced to the equation of the KdV-family, in other words the classical KdV equation (Benjamin 1966), the forced KdV (Grimshaw & Smyth 1986; Zhu, Wu & Yates 1986) and the KdV equation with variable coefficients for uneven bottom (Kakutani 1971; Johnson 1972).

For the flat-bottom case ($h_2 = h_{20}$), the Boussinesq system (A 7)–(A 9) can also be obtained from our new set of equations (4.14)–(4.17) by using the following relationship between the velocity at the interface $\widetilde{\boldsymbol{u}_2}$ and the depth-mean velocity across the lower layer $\overline{\boldsymbol{u}_2}$:

$$\widetilde{\boldsymbol{u}_2} = \overline{\boldsymbol{u}_2} - \tfrac{1}{3}h_{20}{}^2\nabla^2\overline{\boldsymbol{u}_2} + O(\alpha^2\epsilon^2, \alpha\epsilon^4), \tag{A 12}$$

and the shallow water limit for $\boldsymbol{T}\cdot[\widetilde{\boldsymbol{u}_2}]$

$$\boldsymbol{T}\cdot[\widetilde{\boldsymbol{u}_2}] = -h_{20}(\nabla\cdot\widetilde{\boldsymbol{u}_2}) - \tfrac{1}{3}h_{20}{}^3\nabla^2(\nabla\cdot\widetilde{\boldsymbol{u}_2}) + O(\epsilon^5) \qquad \text{as} \quad h_{20}\to 0. \tag{A 13}$$

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